Lecture 5 Time-Dependent BC

In this lecture we shall learn how to solve the inhomogeneous heat equation

$$u_t - \alpha^2 u_{xx} = h(x,t)$$

with time-dependent BC. To begin with, let us consider the following IBVP problem with time-dependent BC:

$$\text{PDE:} \quad u_t = \alpha^2 u_{xx} \quad 0 \le x \le L,\ 0 < t < \infty, \tag{1}$$

$$\text{BC:} \quad u(0,t) = a(t) \ \ u(L,t) = b(t), \quad 0 < t < \infty, \tag{2}$$

$$\text{IC:} \quad u(x,0) = f(x). \tag{3}$$

In the previous lecture, we had discussed the solution of this problem in the case where $a(t)$ and $b(t)$ are constant functions (independent of t) and $f(x)$ is a suitable given function. Notice that the function $w(x,t)$ defined by

$$w(x,t) = \left[\frac{b(t) - a(t)}{L}\right] x + a(t)$$

satisfies the *BC* (2). However, $w(x,t)$ will not satisfy the PDE (1) unless $a(t)$ and $b(t)$ are constant. In fact,

$$w_t - \alpha^2 w_{xx} = \left[\frac{b'(t) - a'(t)}{L}\right] x + a'(t).$$

We now attempt to find a solution for the problem (1)-(3) of the form

$$u(x,t) = w(x,t) + v(x,t),$$

where $v(x,t)$ satisfies the following problem

$$\begin{aligned} v_t - \alpha^2 v_{xx} &= u_t - \alpha^2 u_{xx} - (w_t - \alpha^2 w_{xx}) \\ &= -(w_t - \alpha^2 w_{xx}) \\ &= -[b'(t) - a'(t)]x/L - a'(t). \end{aligned}$$

Further,

$$v(0,t) = u(0,t) - w(0,t) = a(t) - a(t) = 0,$$
$$v(L,t) = b(t) - b(t) = 0.$$

Thus, the function $v(x,t)$ must satisfy the following related problem with homogeneous BC, but inhomogeneous PDE:

$$\text{PDE:} \quad v_t - \alpha^2 v_{xx} = -[b'(t) - a'(t)]x/L - a'(t), \quad 0 \le x \le L,\ 0 < t < \infty, \tag{4}$$

$$\text{BC:} \quad v(0,t) = 0,\ \ v(L,t) = 0, \quad 0 < t < \infty, \tag{5}$$

$$\text{IC:} \quad v(x,0) = u(x,0) - w(x,0) = f(x) - [a(0) - b(0)]x/L - a(0). \tag{6}$$

Note: When $a(t)$ and $b(t)$ are constants, the PDE is homogeneous. But, in this case, $v(x,t)$ satisfies nonhomogeneous PDE.

The problem (4)-(6) is a special case of the following general problem:

$$\text{PDE:} \quad v_t - \alpha^2 v_{xx} = h(x,t) \quad 0 \le x \le L,\ 0 < t < \infty, \tag{7}$$

$$\text{BC:} \quad v(0,t) = 0,\ \ v(L,t) = 0, \quad 0 < t < \infty \tag{8}$$

$$\text{IC:} \quad v(x,0) = g(x). \tag{9}$$

The solution procedure to the above problem was given by the French mathematician and physicist Jean-Marie-Constant Duhamel (1797-1872). The method is known as Duhamel's principle.

Suppose u_1 and u_2 are solutions of the following problems:

$$(P1:)\ \begin{array}{ll} \text{PDE:} & (u_1)_t - \alpha^2 (u_1)_{xx} = 0 \\ \text{BC:} & u_1(0,t) = 0,\ \ u_1(L,t) = 0 \\ \text{IC:} & u_1(x,0) = g(x) \end{array} \qquad (P2:)\ \begin{array}{ll} \text{PDE:} & (u_2)_t - \alpha^2 (u_2)_{xx} = h(x,t) \\ \text{BC:} & u_2(0,t) = 0,\ \ u_2(L,t) = 0 \\ \text{IC:} & u_2(x,0) = 0 \end{array} \tag{10}$$

It is easy to check that $v(x,t) = u_1(x,t) + u_2(x,t)$ solves (7)-(9). The solution u_1 to the problem $(P1)$ is known (cf. Lecture 4 in Module 5). It remains only to solve the problem (P2) for u_2.

The above observation has led to the following (cf. [1]).

Theorem 1. *A solution to problem (1)-(3) is given by*

$$u(x,t) = w(x,t) + u_1(x,t) + u_2(x,t),$$

where

$$w(x,t) = \left[\frac{b(t) - a(t)}{L}\right] x + a(t)$$

is the particular solution of the BC and $u_1(x,t)$ *solves (P1) with* $g(x) = f(x) - w(x,0)$ *and* $u_2(x,t)$ *solves (P2) with* $h(x,t) = -(w_t - \alpha^2 w_{xx}) = -[b'(t) - a'(t)]x/L - a'(t)$.

1 Duhamel's principle

The basic idea of Duhamel's principle is to transfer the source term $h(x,t)$ to initial condition of related problems. This is done in the following manner. The function defined by

$$u(x,t) = \int_0^t v(x,t;s)ds$$

is a solution of (7)-(9) provided $v(x,t;s)$ is a solution of the problem

$$\begin{aligned} \text{PDE:} \quad & v_t = \alpha^2 v_{xx}, \quad 0 \le x \le L,\ 0 < t < \infty, && (11)\\ \text{BC:} \quad & v(0,t;s) = 0,\ \ v(L,t;s) = 0, \quad 0 < t < \infty, && (12)\\ \text{IC:} \quad & v(x,s;s) = h(x,s). && (13) \end{aligned}$$

Note that both PDE and BC are homogeneous. We use translation in time

$$u(x,t) = \int_0^t \bar{v}(x,t-s;s)ds$$

to obtain an IC at $t = 0$, instead of $t = s$. Rewriting (11)-(13) in terms of $\bar{v}$, we now reduce the problem to the following associated problem with IC at $t = 0$:

$$\begin{aligned} \text{PDE:} \quad & \bar{v}_t = \alpha^2 \bar{v}_{xx} \quad 0 \le x \le L,\ 0 < t < \infty, && (14)\\ \text{BC:} \quad & \bar{v}(0,t;s) = 0,\ \ \bar{v}(L,t;s) = 0, \quad 0 < t < \infty && (15)\\ \text{IC:} \quad & \bar{v}(x,0;s) = h(x,s). && (16) \end{aligned}$$

To illustrate the procedure let us consider the following example:
Example 2. *Solve*

$$\begin{aligned} \textit{PDE:} \quad & u_t - \alpha^2 u_{xx} = t\sin(x) \quad 0 \le x \le \pi,\ 0 < t < \infty, && (17)\\ \textit{BC:} \quad & u(0,t) = 0,\ \ u(\pi,t) = 0, \quad 0 < t < \infty, && (18)\\ \textit{IC:} \quad & u(x,0) = 0. && (19) \end{aligned}$$

Solution. Here $h(x,t) = t\sin(x)$. We solve the related problem:

$$\begin{aligned} \text{PDE:} \quad & \bar{v}_t = \alpha^2 \bar{v}_{xx}, \quad 0 \le x \le \pi,\ 0 < t < \infty, && (20)\\ \text{BC:} \quad & \bar{v}(0,t;s) = 0,\ \ \bar{v}(\pi,t;s) = 0, \quad 0 < t < \infty, && (21)\\ \text{IC:} \quad & \bar{v}(x,0;s) = h(x,s) = s\sin(x). && (22) \end{aligned}$$

Treating s a constant, we easily obtain $\bar{v}(x,t;s) = se^{-\alpha^2 t}\sin(x)$. Note that

$$\begin{aligned} u(x,t) &= \int_0^t \bar{v}(x,t-s;s)ds = \int_0^t se^{-\alpha^2(t-s)}\sin(x)ds \\ &= e^{-\alpha^2 t}\sin(x)\int_0^t se^{\alpha^2 s}ds = \Big[(\alpha^2)^{-1}t + (\alpha^2)^{-2}(e^{-\alpha^2 t}-1)\Big]\sin(x), \end{aligned}$$

which satisfies (17)-(19).

Theorem 3. (Duhamel's principle, [1]) *Let $h(x,t)$ be a twice continuously differentiable function in $0 \le x \le L$, $t \ge 0$. Assume that, for each $s \ge 0$, the IBVP*

$$\begin{aligned} &\textit{PDE:} && v_t = \alpha^2 v_{xx} \quad 0 \le x \le L,\ 0 < t < \infty, && (23)\\ &\textit{BC:} && v(0,t;s) = 0,\ \ v(L,t;s) = 0, \quad 0 < t < \infty, && (24)\\ &\textit{IC:} && v(x,0;s) = h(x,s). && (25) \end{aligned}$$

has a solution $v(x,t;s)$, where $v(x,t;s)$, $v_t(x,t;s)$ and $v_{xx}(x,t;s)$ are continuous (in all three variables). Then the unique solution of the problem

$$\begin{aligned} &\textit{PDE:} && u_t - \alpha^2 u_{xx} = h(x,t) \quad 0 \le x \le L,\ 0 < t < \infty, && (26)\\ &\textit{BC:} && u(0,t) = 0,\ \ u(L,t) = 0, \quad 0 < t < \infty, && (27)\\ &\textit{IC:} && u(x,0) = 0. && (28) \end{aligned}$$

is given by

$$\boxed{u(x,t) = \int_0^t v(x,t;s)ds.} \qquad (29)$$

Proof. Note that the function $u(x,t)$ defined by

$$u(x,t) = \int_0^t v(x,t;s)ds$$

satisfies the IC $u(x,0) = 0$ and the BC $u(0,t) = u(L,t) = 0$. Observe that $v(x,t;s)$ satisfies the BC (24). Now, with $g(t,s) = v(x,t;s)$, where x fixed, we have

$$\begin{aligned} u_t(x,t) &= v(x,t;t) + \int_0^t v_t(x,t;s)ds \\ &= h(x,t) + \int_0^t \alpha^2 v_{xx}(x,t;s)ds. \end{aligned}$$

Apply Leibniz's rule to obtain

$$u_t(x,t) = h(x,t) + \alpha^2 u_{xx}(x,t).$$

By the hypothese on $v(x,t;s)$, it follows that $u(x,t)$ is in C^2. For the uniqueness, see Theorem 4 (of Lecture 2 of Module 5).

Remark 4. *The solution u in (29) may be written as*

$$u(x,t) = \int_0^t \bar{v}(x, t-s; s)ds$$

where $\bar{v}$ solves (14)-(16).

Example 5. *Solve the IBVP:*

$$\begin{aligned} &u_t - \alpha^2 u_{xx} = t[\sin(2\pi x) + 2x] \quad 0 \le x \le 1,\ 0 < t < \infty, \\ &u(0,t) = 1, \ \ u(1,t) = t^2, \quad 0 < t < \infty, \\ &u(x,0) = 1 + \sin(\pi x) - x. \end{aligned}$$

Solution. The function that satisfies the BC is

$$w(x,t) = (t^2 - 1)x + 1.$$

Then $u(x,t) = w(x,t) + v(x,t)$, where $v(x,t)$ solves the related problem with homogeneous BC:

$$\begin{aligned} v_t - kv_{xx} &= u_t - \alpha^2 u_{xx} - (w_t - \alpha^2 w_{xx}) = t\sin(2\pi x) \\ v(0,t) &= u(0,t) - w(0,t) = 0 \\ v(1,t) &= u(1,t) - w(1,t) = 0 \\ v(x,0) &= u(x,0) - w(x,0) = \sin(\pi x). \end{aligned}$$

Now, $v = u_1 + u_2$, where u_1 and u_2, respectively, solves

$$(a) \quad \begin{aligned} &(u_1)_t - \alpha^2 (u_1)_{xx} = 0 \\ &u_1(0,t) = 0 \ \ u_1(1,t) = 0 \\ &u_1(x,0) = \sin(\pi x) \end{aligned} \qquad (b) \quad \begin{aligned} &(u_2)_t - \alpha^2 (u_2)_{xx} = t\sin(2\pi x) \\ &u_2(0,t) = 0 \ \ u_2(1,t) = 0 \\ &u_2(x,0) = 0. \end{aligned}$$

We know that $u_1(x,t) = e^{-\pi^2\alpha^2 t}\sin(\pi x)$. The function u_2 is found via Duhamel's principle. The solution u_2 is given by

$$u_2(x,t) = \int_0^t \bar{v}(x, t-s; s)ds,$$

where $\bar{v}$ solves the problem

$$\begin{aligned} &\bar{v}_t = \alpha^2 \bar{v}_{xx} \\ &\bar{v}(0,t;s) = 0 \ \ \bar{v}(L,t;s) = 0 \\ &\bar{v}(x,0;s) = s\sin(2\pi x). \end{aligned}$$

We know that $\bar{v}(x,t;s) = se^{-4\pi^2\alpha^2 t}\sin(2\pi x)$. Thus,

$$\begin{aligned}
u_2(x,t) &= \int_0^t s\cdot e^{-4\pi^2\alpha^2(t-s)}\sin(2\pi x)ds \\
&= e^{-4\pi^2\alpha^2 t}\sin(2\pi x)\int_0^t s\cdot e^{4\pi^2\alpha^2 s}ds \\
&= (4\pi^2\alpha^2)^{-2}\left[4\pi^2\alpha^2 t + e^{-4\pi^2\alpha^2 t} - 1\right]\cdot\sin(2\pi x).
\end{aligned}$$

The solution is then given by

$$u(x,t) = w(x,t) + u_1(x,t) + u_2(x,t).$$

Remark 6. *Duhamel's principle is also applicable to problems with PDE* $u_t - \alpha^2 u_{xx} = h(x,t)$ *and homogeneous BC of the forms:*

$$\begin{aligned} u_x(0,t) &= 0 \\ u(L,t) &= 0 \end{aligned}\;;\qquad \begin{aligned} u(0,t) &= 0 \\ u_x(L,t) &= 0 \end{aligned}\;;\qquad \begin{aligned} u_x(0,t) &= 0 \\ u_x(L,t) &= 0. \end{aligned}\;.$$

Practice Problems

1. Solve the following IBVP:

$$\begin{aligned}
&u_t = \alpha^2 u_{xx} + \cos(3t),\;\; 0 < x < 1,\;\; t > 0, \\
&u_x(0,t) = 0,\; u_x(1,t) = 1,\quad t > 0, \\
&u(x,0) = \cos(\pi x)\frac{1}{2}x^2 - x,\;\; 0 < x < 1.
\end{aligned}$$

2. Solve the following IBVP:

$$\begin{aligned}
&u_t = 4u_{xx} + e^t\sin(x/2) - \sin(t),\;\; 0 < x < \pi,\;\; t > 0, \\
&u(0,t) = \cos(t),\; u(\pi,t) = 0,\quad t > 0, \\
&u(x,0) = 1,\;\; 0 < x < \pi.
\end{aligned}$$